Geometric Optimal Control with Applications

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Chapter 2

Controllability

Higher order necessary optimality conditions for singular extremals are due to Goh, Kelley, and others; see the survey article [10]. The higher order maximum principle is due to Krener [21] and Hermes [15]. The enlargement technique exists in a heuristic form in Hirschorn [16], but was conceptualized by Jurdjevic-Kupka [18] and fully exploited to get controllability conditions for right-invariant systems on Lie groups. The problem of rigidity is everywhere-present in the literature about the abnormal problem in calculus of variations, see for instance Bliss [3]. It also appears in the article [4] in control theory and the neat analysis concerning this problem is due to [1].

Lie brackets play a crucial role in analyzing the controllability properties of nonlinear control systems, and the regularity properties of optimal trajectories.

2.1 Notation from Differential Geometry

We denote by M a smooth (C^∞ or C^ω) manifold of dimension n, connected and second countable. We denote by TM the fiber bundle and by T^*M the cotangent bundle. Let $V(M)$ be the set of smooth vector fields on M and $Diff(M)$ the set of smooth diffeomorphisms.

Definition 1. *Let $X \in V(M)$ and let f be a smooth function on M. The Lie derivative is defined as: $L_X f = df(X)$. If $X, Y \in V(M)$, the Lie bracket is given by*

$$ad\, X(Y) = [X, Y] = L_Y \circ L_X - L_X \circ L_Y.$$

If $x = (x^1, \cdots, x^n)$ are a local system of coordinates we have:

$$X(x) = \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x^i}$$

$$L_X f(x) = \frac{\partial f}{\partial x} X(x)$$

$$[X, Y](x) = \frac{\partial X}{\partial x}(x) Y(x) - \frac{\partial Y}{\partial x}(x) X(x)$$

The mapping $(X, Y) \mapsto [X, Y]$ is $\mathbb{R}$-linear and skew-symmetric. Moreover, the Jacobi identity holds:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Definition 2. *Let $X \in V(M)$. We denote by $x(t, x_0)$ the maximal solution of the Cauchy problem $\dot{x}(t) = X(x(t))$, $x(0) = x_0$. This solution is defined on a maximal open interval J containing 0. We denote by $\exp tX$ the local one parameter group associated to X, that is: $\exp tX(x_0) = x(t, x_0)$. The vector field X is said to be complete if the trajectories can be extended over $\mathbb{R}$.*

Definition 3. *Let $X \in V(M)$ and $\varphi \in Diff(M)$. The image of X by φ is $\varphi * X = d\varphi(X \circ \varphi^{-1})$.*

We recall the following results.

Proposition 1. *if $X, Y \in V(M)$ and $\varphi \in Diff(M)$, we have:*

1. *The one parameter local group of $Z = -\varphi * X$ is given by:*

$$\exp tZ = \varphi \circ \exp tX \circ \varphi^{-1}$$

2. $\varphi * [X, Y] = [\varphi * X, \varphi * Y]$

3. *The Baker-Campbell-Hausdorf (BCH) formula is:*

$$\exp sX \exp tY = \exp \zeta(X, Y)$$

where $\zeta(X, Y)$ belongs to the Lie algebra generated by $[X, Y]$ with:

$$\zeta(X, Y) = sX + tY + \frac{st}{2}[X, Y] + \frac{st^2}{12}[[X, Y], Y] - \frac{s^2t}{12}[[X, Y], X]$$
$$-\frac{s^2t^2}{24}[X, [Y, [X, Y]]] + \cdots ,$$

the series converging for s, t small enough in the analytic case.

4. *We have*

$$\exp tX \exp \varepsilon Y \exp -tX = \exp \eta(X, Y)$$

with $\eta(X, Y) = \varepsilon \sum_{k\geq 0} \frac{t^k}{k!} ad^k X(Y)$ and the series converging for ε, t small enough in the analytic case.

5. *The ad-formula is:*

$$\exp tX * Y = \sum_{k\geq 0} \frac{t^k}{k!} ad^k X(Y)$$

where the series is converging for t small enough.

Definition 4. *A polysystem D is a family $\{V_i; i \in I\}$ of vector fields. We denote by the same letter the associated distribution, that is the mapping $x \mapsto Span\{V(x); V \in D\}$. The distribution D is said to be involutive if $[V_i, V_j] \subset D, \forall V_i, V_j \in D$.*

Definition 5. *Let D be a polysystem. We design by D_{AL} the Lie algebra generated by D. By construction the associated distribution D_{AL} is involutive. The Lie algebra D_{AL} is constructed recursively as follows:*

$$D_1 = Span\{D\},$$
$$D_2 = Span\{D_1 + [D_1, D_1]\},$$
$$\ldots,$$
$$D_k = Span\{D_{k-1} + [D_1, D_{k-1}]\}$$

and $D_{AL} = \cup_{k\geq 1} D_k$. If $x \in M$, we associate the following sequence of integers: $n_k(x) = dim\, D_k(x)$.

2.1.1 Controllability with Piecewise Constant Controls

Definition 6. *Consider a smooth system on M, given in local coordinates by*

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in M, \quad u(t) \in U \subseteq \mathbb{R}^m \tag{2.1}$$

The set of admissable controls $u(\cdot)$ is the set $\mathcal{U}$ of piecewise constant mappings. If $x(t, x_0, u)$ is the solution of 2.1 associated to $u(\cdot)$ starting at at $x(0) = x_0$, we denote by $A(x_0, T)$ the accessibility set $\cup_{u(\cdot)\in\mathcal{U}} x(T, x_0, u)$ in time T and $A(x_0)$ the accessibility set $\cup_{u(\cdot)\in\mathcal{U}} x(T, x_0)$. The system is controllable if for each $x_0 \in M$ we have $A(x_0) = M$.

Example 1. *Consider the problem of a car parallel parking. The state of the car is given by its position and rotation:* $q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \in \mathbb{R}^2 \times S^1$. *The car can drive forward and backward at a bounded speed and can also turn. This allows two controls* u, v *with* $(u, v) \in [-1, 1] \times [-1, 1]$ *to determine the possible trajectories of the vehicle. The system can thus be described as:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v \tag{2.2}$$

Denote $F_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, F_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *and* $F_3 = [F_1, F_2]$. *The computation of* F_3 *is as follows.*

$$\begin{aligned} F_3 = [F_1, F_2] &= \frac{\partial F_1}{\partial q} F_2 - \frac{\partial F_2}{\partial q} F_1 \\ &= \begin{pmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & \sin\theta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \end{aligned}$$

As the rank of $D_{AL}(F_1, F_2, F_3)(q) = 3$, *which is the same as the dimension of the manifold in which q resides* $(\mathbb{R}^2 \times S^1)$, *we will see that this system is controllable. This is due to the Lie bracket allowing movement in a new direction* (F_3).

Definition 7. *Consider a control system 2.1 on* M. *We can associate to this system the polysystem* $D = \{f(\cdot, u); u \text{ constant }, u \in \mathcal{U}\}$. *We denote by* $S_T(D)$ *the set*

$$S_T(D) = \{\exp t_1 V_1 \cdots \exp t_k V_k;\ k \in \mathbb{N}, t_i \geq 0 \text{ and } \sum_{i=1}^{k} t_i = T, V_i \in D\}$$

and by $S(D)$ *the local semi-group:* $\cup_{T \geq 0} S_T(D)$. *We denote by* $G(D)$ *the local group generated by* $S(D)$, *that is*

$$G(D) = \{\exp t_1 V_1 \cdots \exp t_k V_k;\ k \in \mathbb{N}, t_i \in \mathbb{R}, V_i \in D\}.$$

Properties.

1. The accessibility set from x_0 in time T is:
$$A(x_0, T) = S_T(D)(x_0).$$
2. The accessibility set from x_0 is the orbit of the local semi-group:
$$A(x_0) = S(D)(x_0).$$

Definition 8. *We call the orbit of* x_0 *the set* $O(x_0) = G(D)(x_0)$. *The system is said to be weakly controllable if for every* $x_0 \in M$, $O(x_0) = M$.

2.1.2 Integrating Distributions

Let D be a polysystem and D_{AL} the Lie algebra generated by D. We consider the distribution $\Delta : x \mapsto D_{AL}$. It is an involutive distribution and the problem of integrating Δ at a point x_0 is to find a submanifold N, containing x_0 such thtat for each $y \in N$, $T_y N = \Delta(y)$. It is a generalization of the Cauchy problem for integrating a single vector field. Here, we are presenting two results:

1. If near x_0 the rank of Δ is constant, then we have the **Frobenius Theorem** which is a generalization of the theorem of linearization of a smooth vector field X near a regular point.
2. If the rank is not constant but if the vector fields of the polysystem are real analytic, then the result is still true. It was proved by Nagano-Sussmann.

The proofs of both are radically different.

2.1.3 Frobenius Theorem

Theorem 1. ***(Frobenius Theorem)*** *Assume that the rank of the distribution Δ is constant near the point $x_0 : rank\Delta = p$. Then there exists a local coordinate system $x = (x^1, \cdots, x^n)$ such that Δ is generated by $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$. In particular near x_0 the integral manifolds are given by x^i constant, for $i = p+1, \cdots, n$.*

Proof. The proof is standard and is a reccurence on p.

If $p = 1$, then locally $\Delta = \mathbb{R}X$ where $X \in V(M), X(x_0) \neq 0$. We use the linearization theorem for ordinary differential equations.

If $p \geq 2$, then locally $\Delta = Span\{Y_1, \cdots, Y_p\}$. We choose a coordinate system $y = (y^1, \cdots, y^n)$ centered at x_0 such that $Y_1 = \frac{\partial}{\partial y^1}$. Consider the following p vector fields $Z_1, \cdots, Z_p$ of Δ defined by

$$Z_1 = Y_1, \qquad Z_k = Y_k - (L_{Y_k} y^1)Y_1,\ 2 \leq k \leq p.$$

By construction $L_{Z_k} y^1 = 0$ for $2 \leq k \leq p$. Hence locally Z_k is tangent for $2 \leq k \leq p$ to the submanifold $S : y^1 = 0$. Therefore there exist $(p-1)$ vector fields V_k on S which are the restriction of the vector fields Z_k to S. They define on S an involutive distribution $\tilde{\Delta}$ of rank $(p-1)$. We use the reccurence assumption which asserts that there exists on S a local coordinate system $z = (z^1, \cdots, z^n)$ such that

$$\tilde{\Delta} = Span\{V_2, \cdots, V_k\} = Span\{\frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^p}\}.$$

We can define a local coordinate system centered at x_0 by

$$x^1 = y^1, \quad x^i = z^i, \quad 2 \leq i \leq n.$$

We claim that $\Delta = Span\{Z_1, \cdots, Z_p\}$ coincides locally with the flat distribution $Span\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$. To prove it, it is sufficient to check that

$$L_{Z_k} x^{p+r} = 0 \text{ for } r \geq 1 \text{ and } 1 \leq k \leq p.$$

1. For $k = 1$: since $Z_1 = \frac{\partial}{\partial x^1}$, we have $L_{Z_1} x^i = 0$ for $i \geq 2$.

2. For $k \geq 2$: first of all we observe that

$$\frac{\partial}{\partial x^1}(L_{Z_k} x^{p+r}) = L_{Z_1}(L_{Z_k} x^{p+r}).$$

   Since $L_{Z_1} x^{p+r} = 0$, we can write

$$\frac{\partial}{\partial x^1}(L_{Z_k} x^{p+r}) = L_{[Z_k, Z_1]}(x^{p+r})$$

   and we know by construction that

$$[Z_1, Z_k] \in Span\{Z_j; j \geq 2\}.$$

   Hence we can write

$$\frac{\partial}{\partial x^1}(L_{Z_k} x^{p+r}) = \sum_{j=2}^{p} \lambda_j (L_{Z_j} x^{p+r})$$

   where the λ_j are scalar. It is a linear differential equation with respect to x^1. For $x^1 = 0$, we have $Z_k = V_k$ and by construction $L_{V_k} x^{p+r} = 0$. Since the solution of a linear system with values 0 at $x^1 = 0$ is the identically zero solution, we have $L_{Z_k} x^{p+r} = 0$.

□

2.1.4 Nagano-Sussman Theorem

See [30]. When the rank condition is satisfied (rankΔ = constant) we get from the Frobenius theorem a description of all the integral manifolds near x_0. If we only need to construct the leaf passing through x_0 the rank condition is clearly too strong. Indeed, if $D = \{X\}$ is generated by a single vector field X, there exists an integral curve through x_0 which is locally Lipschitz. For a family of vector fields this result is still tru if the vector fields are analytic.

Theorem 2. ***(Nagano-Sussman Theorem)*** *Let D be a family of analytic vector fields near $x_0 \in M$ and let p be the rank of $\Delta : x \mapsto D_{AL}(x)$ at x_0. Then through x_0 there exists locally an integral manifold of dimension p.*

Proof. Let p be the rank of Δ at x_0. Then there exists p vector fields of $D_{AL} : X_1, \cdots, X_p$ such that $Span\{X_1(x_0), \cdots, X_p(x_0)\} = \Delta(x_0)$. Consider the mapping

$$\alpha : (t_1, \cdots, t_p) \mapsto \exp t_1 X_1 \cdots \exp t_p X_p(x_0).$$

It is an immersion for $(t_1, \cdots, t_p) = (0, \cdots, 0)$. Hence the image denoted by N is locally a submanifold of dimension p. To prove that N is an integral manifold we must check that for each $y \in N$ near x_0, we have $T_yN = \Delta(y)$. It is a direct consequence of the equalities

$$D_{AL}(\exp tX_i(x)) = d\exp tX_i(D_{AL}(x)),\ i = 1, \cdots, p$$

and x near x_0, t small enough. To show that the previous equalities hold, let $V(x) \in D_{AL}(x)$ such that $V(x) = Y(x)$. By analycity and the ad-formula for t small enough we have

$$(d\exp tX_i)(Y(x)) = \sum_{k\geq 0} \frac{t^k}{k!} ad^k X_i(Y)(\exp tX_i(x)).$$

Hence for t small enough, we have

$$(d\exp tX_i)(D_{AL}(x)) \subset D_{AL}(\exp tX_i(x)).$$

Changing t to $-t$ we show the second inclusion. □

2.1.5 C^∞-Counter Example

To prove the previous theorem we use the following geometric property. Let X, Y be two analytic vector fields and assume $X(x_0) \neq 0$. From the ad-formula, if all the vector fields are $ad^k X(Y), k \geq 0$ are collinear to X at x_0, then for t small enough the vector field Y is tangent to the integral curve $\exp tX(x_0)$.

Hence is is easy to construct a C^∞-counter example using flat C^∞-mappings. Indeed take $f : \mathbb{R} \mapsto \mathbb{R}$ a smooth mapping such that $f(x) = 0$ for $x \leq 0$ and $f(x) \neq 0$ for $x > 0$. Consider the two vector fields on $\mathbb{R}^2 : X = \frac{\partial}{\partial x}$ and $Y = f(x)\frac{\partial}{\partial y}$. At 0, D_{AL} is of rank 1. Indeed, we have $[X,Y](x) = -f'(x)\frac{\partial}{\partial y} = 0$ at 0 and hence $[X,Y](0) = 0$. The same is true for all high order Lie brackets. In this example the rank D_{AL} is not constant along $\exp tX(0)$, indeed for $x > 0$, the vector field Y is transverse to this vector field.

2.1.6 Nonlinear Controllability and Chow Theorem

Theorem 3. ***(Chow).*** *Let D be a C^∞-polysystem on M. We assume that for each $x \in M, D_{AL}(x = T_xM)$. Then we have*

$$G(D)(x) = G(D_{AL}(x)) = M,$$

for each $x \in M$.

Proof. Since M is connected it is sufficient to prove the result locally. The proof is based on the BHC-formula. We assume $M = \mathbb{R}^3$ and $D = \{X, Y\}$ with rank $\{X, Y, [X,Y]\} = 3$ at x_0; the generalization is straightforward. Let λ be a real number and consider the mapping

$$\varphi_\lambda : (t_1, t_2, t_3) \mapsto \exp \lambda X \exp t_3 Y \exp -\lambda X \exp t_2 Y \exp t_1 X(x_0).$$

We prove that for λ small but nonzero, φ_λ is an immersion. Indeed using the BHC formula we have

$$\varphi_\lambda(t_1,t_2,t_3) = \exp(t_1X + (t_2+t_3)Y + \frac{\lambda t_3}{2}[X,Y] + \cdots)(x_0),$$

hence

$$\frac{\partial\varphi_\lambda}{\partial t_1}(0,0,0) = X(x_0), \qquad \frac{\partial\varphi_\lambda}{\partial t_2}(0,0,0) = Y(x_0),$$
$$\frac{\partial\varphi_\lambda}{\partial t_3}(0,0,0) = Y(x_0) + \frac{\lambda}{2}[X,Y](x_0) + o(\lambda)$$

Since $X, Y, [X,Y]$ are linearly independent at x_0, the rank of φ_λ at 0 is 3 for $\lambda \neq 0$ small enough. □

Definition 9. *The polysystem D is called weakly controllable if the orbit $O(x)$ of $G(D)$ is M for every $x \in M$. The polysystem D is called controllable if the orbit $A(x)$ of $S(D)$ is M for every $x \in M$. The polysystem D is said symmetric if for every $X \in D$, we have $-X \in D$.*

Example 2. *Let $D = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ on $\mathbb{R}^2$. Hence $O(0) = \mathbb{R}^2$ and $A(0) = \{x \geq 0, y \geq 0\}$. In general we have that $A(x)$ is a strict subset of $O(x)$.*

Corollary 1. *Let D be a symmetric polysystem. Assume that*

$$\text{rank } D_{AL} = n \ (\dim M) \text{ for every } x.$$

Then D is controllable. In the analytic case this rank condition is also necessary.

Proof. Since D is symmetric, we have: *orbit* $S(D) =$ *orbit* $G(D)$. Then we apply the Chow theorem. In the analytic case, we apply the Nagano-Sussmann theorem. □

The symmetric case is the only case where we can conclude trivially that $S(D) = G(D)$. In general the problem to decide if a semi-group is transitive is difficult.

Nevertheless the following weaker result is true, see [31].

Proposition 2. *Let D be a polysystem. If $\dim D_{AL} = n$ ($\dim M$) for every $x \in M$ then for each neighborhood V of x there exists a nonempty open set U contained in $V \cap A(x)$.*

Proof. Let $x \in M$. If $\dim M \geq 1$, there exists $X_1 \in D$ such that $X_1 \neq 0$, otherwise we would have that $\dim D_{AL}(x) = 0$. Consider the integral curve $\alpha_1 : t \mapsto \exp tX_1(x)$. If $\dim M \geq 2$, then there exists in every neighborhood V of x a point $y \in M$ such that $y = \exp t_1X_1(x)$ and vector field $X_2 \in D$ such that X_2 and X_1 are not collinear at y, otherwise we would have $\dim D_{AL} = 1$. Consider the mapping $\alpha_2 : (t_1,t_2) \mapsto \exp t_2X_2 \exp t_1X_1(x)$. If $\dim M \geq 3$, then there exists in every neighborhood of y a vector field X_3 transverse to the image α_2. With this method we construct in every neighborhood V of x a mapping $\alpha_n : (t_1,\cdots,t_n) \mapsto \exp t_nX_n \cdots \exp t_1X_1(x)$ such that in a point $z = \alpha_n(t_1^*,\cdots,t_n^*)$ of V, α_n is an immersion. This construction provides a nonempty open set U sontained in $V \cap O(x)$. □

2.1.7 Poisson Stability and Controllability

Definition 10. *Let X be a C^∞-vector field on M. The point $x_0 \in M$ is said Poisson stable if for every $T > 0$ and every neighborhood V of x_0 then there exists $t_1, t_2 \geq T$ such that $\exp t_1X(x_0)$ and $\exp -t_2X(x_0) \in V$. The vector field X is called Poisson stable if the set of Poisson stable points is dense in M.*

Theorem 4. ***(Poincaré) [12]*** *If M is a compact manifold with a volume form ω, each conservative vector field X is Poisson stable.*

Proposition 3. *Let D be a polysystem. Assume the following:*

i) for every $x \in M$, rank $D_{AL}(x) = n$ ($\dim M$);

ii) every vector field $X \in D$ is Poisson stable.

Then the system is controllable.

Proof. Here is an outline of the proof, see [22] for details. Let $x, y \in M$, we must show that there exists $X_1, \cdots, X_k \in D$ and $t_1, \cdots, t_k > 0$ such that

$$y = \exp t_1 X_1 \cdots \exp t_k X_k(x).$$

Since D satisfies the rank condition, we can apply Proposition 2 to D and $-D$ to find the existence of two points x', y' and two open sets U and V such that $x' \in U, y' \in V$ such that x' can be steered using $X_i \in D, -D$ to each point of U and each point of V can be steered to y'. To prove the proposition it is sufficient to show that there exist two points $x'' \in U$ and $y'' \in V$ such that x'' can be steered to y''. Since the polysystem satisfies the rank condition, there exist p vector fields $Y_1, \cdots, Y_p$ and p nonzero (positive or negative) real numbers s_i such that

$$y' = \exp s_1 Y_1 \cdots \exp s_p Y_p(x'').$$

In the previous sequence each element $\exp s_k Y_k$ corresponding to the negative time s_k can be nearby replaced by an arc $\exp s'_k Y_k$ using the Poisson-stability of Y_k. The result follows. □

2.1.8 Application

A first application of Proposition 3 is to construct many controllable polysystems on compact manifolds using Poincaré theorem.

Example 3. *Take $M = G$ a compact Lie group and D a polysystem whose each vector field is a right invariant. Then the polysystem is controllable if and only if the rank condition is satisfied. In this case, observe that D_{AL} is a Lie sub-algebra of $g \simeq T_e G$ and hence is finite dimensional. There exist algorithms to compute D_{AL}.*

2.1.9 Controllability and Enlargement Technique

Let D be a polysystem satisfying the rank condition: rank $D_{AL}(x) = dim\, M$ for all $x \in M$. To study the controllability of such a polysystem, a powerful technique is the enlargement technique which was codified by Jurdjevic-Kupka [18]. The principle is simple, we enlarge D using operations which are not modifying the controllability of D. We shall briefly explain these operations. See also [17].

Lemma 1. *Let D be a polysystem such that rank $D_{AL}(x) = dim\, M$ for all x. Then the polysystem D is controllable if and only if the adherence of $S(D)(x)$ is M for every $x \in M$.*

Proof. Use Proposition 2. □

Definition 11. *Let D, D' be two polysystems satisfying the rank condition. We say that D and D' are equivalent if for every $x \in M : S(D)(x) = S(D')(x)$. The union of all polysystem D' equivalent to D is called the saturated of D and is denoted by sat D.*

Clearly a polysystem D is controllable if and only if *sat* D is controllable. Now, we define the codified operations.

Proposition 4. *Let D be a polysystem, then the convex cone generated by sat D is equivalent to D.*

Proof. Clearly if $X \in D$ then $\lambda X \in sat\, D$ for every $\lambda > 0$ (reparameterization). Let $X, Y \in D$, using BHC formula we have

$$\prod_{n\ times} \exp \frac{t}{n} X \exp \frac{t}{n} Y = \exp(t(X+Y) + o(\frac{1}{n})).$$

Taking the limit when $n \to +\infty$, we have $X + Y \in sat\, D$. □

Proposition 5. *Let $X \in D$ and assume X Poisson stable, then $-X \in D$.*

Proof. It is a consequence of the proof of Proposition 3. □

Proposition 6. *If $\pm X, \pm Y \in D$, then $\pm[X,Y] \in sat\, D$.*

Proof. Apply the BHC formula. ☐

Example 4. *Consider a control system of form*

$$\dot{x}(t) = F_0(x(t)) + \sum_{i=0}^{m} F_i(x(t))u_i(t)$$

where the $u_i(t)$ are piecewise constant functions in $\{\pm 1\}$. Let $m = 1, F_0 = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, F_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, resulting in

$$\dot{x}(t) = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

Then, for the polysystem $D = (F_0, F_1)$, the first terms of the Lie algebra D_{AL} are:

$$[F_0, F_1] = \frac{\partial F_0}{\partial x} F_1 - \frac{\partial F_1}{\partial x} F_0 = \begin{pmatrix} 0 & 2x_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 0 \end{pmatrix},$$
$$[[F_0, F_1], F_1] = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Then it is clear that $\dim D_{AL} = 2$, and the rank condition is satisfied. However, consider any trajectory beginning with $x_2(0) > 0$. Then the system is not controllable, as these trajectories can not move in the negative x_1 direction. This shows that the rank condition is necessary but not sufficient for controllability.

Definition 12. *Let D be a polysystem on M. The normalizer $N(D)$ of D is the set of diffeomorphisms φ on M such that for every $x \in M$, $\varphi(x)$ and $\varphi^{-1}(x)$ belong to the adherence of $S(D)(x)$.*

Proposition 7. *Let D be a polysystem, X be an element of D and $\varphi \in N(D)$. Then $\varphi * X$ belongs to $sat\, D$.*

Proof. Let $Y = \varphi * X$, then we have

$$y = (\exp tY)(x) = \varphi \circ \exp tX \circ \varphi^{-1}(x)$$

Hence y belongs to the adherence of $S(D)(x)$. The proposition follows. ☐

Proposition 8. *If D is a polysystem, then the closure of D for the topology of uniform convergence on the compact sets belongs to $sat\, D$.*

Proof. If $X_n \to X$ when $n \to +\infty$ for the above topology, then $\exp tX_n \mapsto \exp tX$ when $n \to +\infty$ on each compact set. ☐

Consider the linear system of $\mathbb{R}^n : \dot{x}(t) = Ax(t) + Bu(t), u(t) \in \mathbb{R}^p, A, B$ constant matrices. Then the linear system is controllable if and only if the rank of $\mathcal{R} = [B, \cdots, A^{n-1}B]$ is n. Indeed, the system can be written $\dot{x} = Ax(t) + \sum_{i=1}^{p} u_i(t)b_i$ and we introduce the polysystem $D = \{Ax + \sum_{i=1}^{p} u_i b_i; u_i \in \mathbb{R}^p\}$. Computing we have

$$D_{AL} = Ax \oplus Im\mathcal{R}$$

and the rank is minimal at 0 and equals to the rank of $\mathcal{R}$. Since the system is analytic, we must have $rank\, \mathcal{R} = n$.

Let us prove the converse. Let $i \in \{1, \cdots, p\}$ then $\pm b_i \in sat\, D$. Indeed for every $n \in N$

$$\frac{1}{n}(Ax + Bu) \in sat\, D$$

and setting $u_i = n\varepsilon, \varepsilon = \pm 1$ we have

$$\lim_{n \to +\infty} \frac{1}{n}(A + n\varepsilon b_i) = \varepsilon b_i \in sat\, D.$$

Hence for every $\lambda \in \mathbb{R}$, we have
$$\exp \lambda b_i * Ax \in sat\ D.$$
Computing we obtain
$$\exp \lambda b_i * Ax = \sum_{k \geq 0} \frac{\lambda^k}{k!} ad^k b_i(Ax)$$
and $ad^k b_i(Ax) = 0$ for $k \geq 2$. Hence for $\lambda \neq 0$ we have
$$\frac{1}{|\lambda|} \exp \lambda b_i * Ax = \frac{1}{|\lambda|}(Ax - \lambda A b_i) \in sat\ D.$$
Taking the limit when $\lambda \to \infty$ we obtain $\pm A b_i \in sat\, D$. Then we repeat the same operation, replacing b_i by Ab_i. At the end we have
$$Span\{b_i, \cdots, A^k b_i, \cdots; i = 1, \cdots, p\} = \mathcal{R} \in sat\ D$$
The result is proved.

Proposition 9. *Consider the following affine system on M :*
$$\dot{x}(t) = F_0(x(t)) + \sum_{i=1}^{p} u_i(t) F_i(x(t)), \qquad u_i(t) \in \mathbb{R}.$$
Let D be the distribution
$$x \mapsto Span\{F_1(x), \cdots, F_p(x)\}.$$
If rank $D_{AL} = dim\, M$, for all $x \in M$, then the system is controllable.

Proof. As before for every $n \in N, u_i = n\varepsilon, \varepsilon = \pm 1, u_j = 0$ if $j \neq i$
$$\frac{1}{n}(F_0 + n\varepsilon F_i) \in sat\ D$$
and by making $n \to +\infty$ we have $\pm F_i \in sat\, D$ for $i = 1, \cdots, p$. Applying Proposition 6 we have $D_{AL} \in sat\, D$. If D_{AL} is of rank n, the system is controllable. □

2.1.10 Evaluation of the Accessibility Set

The Baker-Campbell-Hausdorff formula can be used to make evaluation the accessibility set by constructing an approximation cone. In spirit it is similar to the idea pf Pontryagin maximum principle where we construct the first order Pontryagin cone. This was extensively used by Hermes [15] to get higher order necessary optimality conditions along a singular arc.

Definition 13. *A rational polynomial is an expression of the form $\sum_{i=1}^{l} c_i t^{q_i}$, where $l \in \mathbb{N}, t > 0$ small enough, $q_i \in \mathbb{Q}$ and $c_i \in \mathbb{R}$. It is called positive if $c_i \geq 0$ for all $i = 1, \cdots, l$. Let $X, Y \in V(M), D$ be thepolysystem $\{X, Y\}$, D_{AL} the Lie algebra generated by D. We denote by $\mathcal{E}$ the set of germs of vector fields W such that there exists $k \in \mathbb{N}$ and rational polynomials $r_, \cdots, r_k, s_1, \cdots, s_k : [0, \varepsilon] \to \mathbb{R}$ such that*
$$\exp r_k(t) X \exp s_k(t) Y \cdots \exp r_1(t) X \exp s_1(t) Y = \exp(tW + O(t^\alpha))$$
with $\alpha > 1$.

From the BHC formula, the set $\mathcal{E}$ is contained in D_{AL}. We shall prove the following result.

Theorem 5. *The set $\mathcal{E}$ is D_{AL}*

In order to prove this result we need several lemmas and the following two formulae

$$\exp t_1 X \exp t_2 W \exp -t_1 X = \exp\left(t_2(\sum_{k=0}^{n} \frac{t_1^k}{k!} ad^k X(W) + o(t_1^n))\right) \tag{2.3}$$

$$\begin{aligned}\exp t_1 X \exp t_2 W = \exp(t_1 X + t_2 W + \frac{t_1 t_2}{2}[X,W] + \frac{t_1 t_2^2}{12}[[X,W],W] \\ - \frac{t_1^2 t_2}{12}[[X,W],X] - \frac{t_1^2 t_2^2}{24}[X,[W,[X,w]]] + \cdots)\end{aligned} \tag{2.4}$$

Lemma 2. *The set $\mathcal{E}$ is convex.*

Proof. Let $\lambda \in [0,1]$ and $W_1, W_2 \in \mathcal{E}$. Then there exists $k_1, k_2 \in \mathbb{N}$ and rational polynomials such that

$$\exp r_{k_1}^1(t)X \cdots \exp s_1^1(t)Y = \exp(tW_1 + O(t^\alpha))$$

$$\exp r_{k_2}^2(t)X \cdots \exp s_1^2(t)Y = \exp(tW_2 + O(t^\alpha))$$

Hence we have

$$\begin{aligned}\exp(r_{k_1}^1(\lambda t)X) \cdots \exp(s_1^1(\lambda t)Y) \exp(r_{k_2}^2(1-\lambda)t)X \cdots \exp(s_1^2(1-\lambda)t)Y \\ = \exp(\lambda t W_1 + (1-\lambda)tW_2 + O(t^\alpha))\end{aligned}$$

where $\alpha > 1$. Hence $\lambda W_1 + (1-\lambda)W_2 \in \mathcal{E}$. □

Lemma 3. *The vector fields $-X$ and $-Y$ belong to $\mathcal{E}$.*

Proof. For $\alpha, \beta \in \mathbb{R}$ we have

$$\exp(\alpha t X)\exp(\beta t Y) = \exp(t(\alpha X + \beta Y) + O(t^2)).$$

If we set $\alpha = -1, \beta = 0$, we have $-X \in \mathcal{E}$, and if we set $\alpha = 0, \beta = -1$, we have $-Y \in \mathcal{E}$. □

Proof. **(Theorem 5).** We must show that if $\pm W \in \mathcal{E}$, then $\pm[X,W]$ and $\pm[Y,W] \in \mathcal{E}$. Since $\pm W \in \mathcal{E}$, there exist rational polynomials such that

$$\begin{aligned}\exp r_k(t)X \cdots \exp s_1(t)Y = \exp(tW + O(t^\alpha)) \\ \exp r_k'(t)X \cdots \exp s_1'(t)Y = \exp(-tW + O(t^\alpha))\end{aligned}$$

with $\alpha > 1$. Let $\beta \in \mathbb{Q}$ with $0 < \beta < 1$ and $\alpha\beta > 1$. Then

$$\begin{aligned}\exp(t^{1-\beta}X)\exp(t^\beta W + O(t^{\alpha\beta}))\exp(-t^{1-\beta}X)\exp(-t^\beta W + O(t^{\alpha\beta})) \\ = \exp(t^\beta W + t[X,W] + \cdots)\exp(-t^\beta W + O(t^{\alpha\beta})) \\ = \exp(t[X,W] + \cdots).\end{aligned}$$

This proves that $[X,W] \in \mathcal{E}$. Hence by reccurrence we show that

$$\pm[ad^{k_n}Y, [ad^{k_{n-1}}X, [\cdots, [ad^{k_1}X, Y]\cdots]$$

belongs to $\mathcal{E}$. Since these Lie brackets are generating D_{AL} the theorem is proved. □

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